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Limit of Classical Projections of Quantum Mechanics as $\hbar \to 0$

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The convergence of the Hamiltonians of classical projections to the Hamiltonian of the classical limit is investigated. The convergence of dynamics is shown for Hamiltonians generated by a certain class of functions, in particular by functions from the Schwartz space.

1. INTRODUCTION

The relationship between classical mechanics (CM) and quantum mechanics (QM) has been investigated from the very beginning of QM. QM is characterized by a universal physical constant (Planck constant $\hbar$) which is so small that it can be practically neglected with respect to usual classical effects. This fact gives us one of the keys for the relation between QM and CM. In a sense, CM should be a limit of QM for $\hbar \to 0$. Hepp (1974) proved that the time evolution of the usual classical limit can be obtained as a limit (for $\hbar \to 0$) of the time evolutions of the mean values of position and momentum of some quantum systems parametrized by the value of $\hbar$ where the initial states of these systems are conveniently chosen on the orbit of coherent states.

The Hamiltonian of the usual classical limit can be obtained as a limit (for $\hbar \to 0$) of the Hamiltonians of the "classical projections of QM" parametrized by the value of $\hbar$ (Bóna, 1983). The theory of classical projections was developed, e.g., by Bóna (1986). In the work of Bóna (1983), this limit is investigated for the usual Hamiltonian with potential energy, and this assertion is indicated for another form of the Hamiltonians. A general method

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for finding classical limits in arbitrary quantum theories is developed in the work of Yaffe (1982). Some of the results mentioned there are similar, but not identical, to those presented in the present paper.

Here we shall investigate the limit $\hbar \to 0$ for a certain class of Hamiltonians (Section 2). In fact we prove more, namely the uniform convergence. For Hamiltonians generated by a certain class of functions (in particular by functions from the Schwartz space) we prove even more, namely the convergence of the time evolutions of the classical projections to the time evolution of the classical limit for $\hbar \to 0$ uniformly on compacts (Section 3). This can be considered to be the main contribution of the present paper.

2. THE CONVERGENCE OF HAMILTONIANS

Let U be a unitary irreducible representation of the Weyl–Heisenberg group in a Hilbert space $\mathcal{H}$. Its generators $X_0, \ldots, X_{2n}$ can be chosen so that the canonical commutation relations (CCR) are satisfied:

$$X_0 = \hbar I$$

$$X_i = Q_i, \qquad i = 1, \ldots, n$$

$$X_i = P_{i-n}, \qquad i = n + 1, \ldots, 2n$$

The CCR can be realized in Schrödinger form. Then

$$Q_j \varphi(q) = q_j \varphi(q)$$

$$P_j \varphi(q) = -i\hbar \frac{\partial}{\partial q_j} \varphi(q)$$

The representation U can be considered as a projective representation of the additive group R^{2n} and

$$U_x = \exp\left(\frac{i}{\hbar} X \cdot S \cdot x\right)$$

where $X \cdot S \cdot x = X_j S_{jk} x_k$ and S is the standard symplectic $2n \times 2n$ matrix with elements S_{jk} defined by

$$S_{jj+n} = -S_{j+nj} = 1, \quad j = 1, 2, \ldots, n, \qquad S_{jk} = 0 \quad \text{otherwise}$$

and

$$X = (X_1, \ldots, X_{2n})$$

$$x = (x_1, \ldots, x_{2n}) = (q, p)$$

where

$$q = (q_1, \ldots, q_n)$$
$$p = (p_1, \ldots, p_n)$$

so $x \in R^{2n}$. Let now the Planck constant to approach zero, so instead of $\hbar$ we write $\lambda^2\hbar$, $\lambda \to 0$, everywhere. The generators of the corresponding representation of CCR will be chosen as

$$Q_j^\lambda \varphi(q) = \lambda q_j \varphi(q)$$
$$P_j^\lambda \varphi(q) = -i\lambda\hbar \frac{\partial}{\partial q_j} \varphi(q)$$

which means

$$Q_j^\lambda = \lambda Q_j, \qquad P_j^\lambda = \lambda P_j$$

The corresponding representation will appear as

$$U_x^\lambda = \exp\left(\frac{i}{\lambda^2\hbar} X^\lambda \cdot S \cdot x\right) = \exp\left(\frac{i}{\lambda\hbar} X \cdot S \cdot x\right) = \exp\left(\frac{i}{\hbar} X \cdot S \cdot \frac{x}{\lambda}\right) = U_{(x/\lambda)}$$

Now let ψ be a convenient analytic vector of the representation U. The orbit $O_\psi^\lambda = U^\lambda(G)\psi$ will be a symplectic manifold parametrized by a parameter $x \in R^{2n}$, and diffeomorphic to R^{2n}. We can consider a classical Hamiltonian

$$h_\lambda(x) = (U_x^\lambda \psi, H^\lambda U_x^\lambda \psi)$$

on this orbit, where H^λ is a version of formally given quantum Hamiltonian

$$H^\lambda = f(X_1^\lambda, \ldots, X_{2n}^\lambda)$$

where f is a real function. This classical system is called a classical projection. Bóna (1986) mentions that for H^λ polynomial in X_j^λ

$$\lim_{\lambda \to 0} h_\lambda(x) = h(x) := f(x) \tag{1}$$

holds. In the present paper we shall prove this statement for a convenient class of functions f.

The main technical problem is the definition of the operator $f(X_1^\lambda, \ldots, X_{2n}^\lambda)$ for given real function f. The difficulties arise because of the noncommutativity of the operators X_i^λ. We shall use the Weyl method for symmetrization described, e.g., by Berezin and Shubin (1983). It uses the Fourier transform. Using these techniques we shall seek functions f for which (1) will hold.

Let

$$f(q, p) = \int e^{i(rq+sp)}\varphi(r, s)d^n r\, d^n s \tag{2}$$

where

$$r = (r_1, \ldots, r_n), \qquad s = (s_1, \ldots, s_n)$$

and $\varphi(r, s)$ is the Fourier transform of the function f. Let, according to Berezin and Shubin (1983),

$$f(Q^\lambda, P^\lambda) = \int \exp(i(rQ^\lambda + sP^\lambda))\varphi(r, s)d^n r\, d^n s$$

where

$$(Q^\lambda, P^\lambda) = (Q_1^\lambda, \ldots, Q_n^\lambda, P_1^\lambda, \ldots, P_n^\lambda) = (X_1^\lambda, \ldots, X_{2n}^\lambda) = X^\lambda$$

Hence

$$f(Q^\lambda, P^\lambda) = \int e^{i\lambda(rQ+sP)}\varphi(r, s)d^n r\, d^n s$$

We give now the correct definition of this formal expression. Let the operator $f(Q^\lambda, P^\lambda)$ be defined by

$$f(Q^\lambda, P^\lambda)\phi = \int e^{i\lambda(rQ+sP)}\phi\varphi(r, s)d^n r\, d^n s \tag{3}$$

where the right-hand side is defined as the Bochner integral.

The main result of this section is the following:

Theorem 1. Let $\varphi \in L^1(R^{2n})$ and f be given by (2). Then the relation (1) holds, where $H^\lambda = f(Q^\lambda, P^\lambda)$ is a bounded operator determined by (3). The convergence in (1) is the uniform convergence on the whole R^{2n}.

Proof. Let us examine for which vectors ϕ and which functions φ the integral (3) converges. Obviously if it converges, then $\varphi \in L^1(R^{2n})$, because according to a well known theorem (see, e.g., Blank, Exner, Havlíček, 1993, Theorem 3.7.4c) the Bochner integrability of the function $e^{i\lambda(rQ+sP)}\phi\varphi(r, s)$ implies the Lebesgue integrability of the norm

$$\left\| e^{i\lambda(rQ+sP)}\phi\varphi(r, s)\right\| = \|\phi\|\,|\varphi(r, s)|$$

i.e., $\varphi \in L^1(R^{2n})$. It is well known that the integral converges for $\varphi \in C_0^\infty(R^{2n})$ for all $\phi \in \mathcal{H}$.

We show that for functions $\varphi \in L^1(R^{2n})$ this Bochner integral converges for arbitrary vector ϕ. According to the Lebesgue dominated convergence theorem, a function $\varphi \in L^1(R^{2n})$ is integrable. The function $(r, s) \mapsto e^{i\lambda(rQ+sP)}\phi$ is continuous (from the Weyl relations), so it is uniformly continuous on compacts. We construct a sequence of simple functions S_k: $R^{2n} \to \mathcal{H}$ such that pointwise $S_k \to e^{i\lambda(rQ+sP)}\phi$. For arbitrary $k \in N$ consider a $2n$-dimensional cube $C_k := \langle -k, k\rangle^{2n}$. Because of the uniform continuity on compacts, for arbitrary k there exists $\delta_k > 0$ such that if the Euclidean distance between the points (r, s) and (r', s') of R^{2n} is less than δ_k, then

$$\left\| e^{i\lambda(rQ+sP)}\phi - e^{i\lambda(r'Q+s'P)}\phi \right\| < \frac{1}{k}$$

The cube C_k can be considered as the union

$$C_k = \bigcup_{j=1}^{N} A_j$$

of sets with diameters less than δ_k. For each $j \in \{1, \ldots, N\}$ we choose arbitrary $(r, s) \in A_j$ fixed and for each $(r', s') \in A_j$ we put

$$S_k(r', s') := e^{i\lambda(rQ+sP)}\phi$$

This defines the function S_k on C_k. On the exterior of C_k we simply put $S_k \equiv 0$. So we constructed a simple function S_k such that for $(r, s) \in C_k$

$$\left\| S_k(r, s) - e^{i\lambda(rQ+sP)}\phi \right\| < \frac{1}{k}$$

This implies the pointwise convergence

$$S_k \to e^{i\lambda(rQ+sP)}\phi$$

In the same way, for the Lebesgue integrable function φ there exists a sequence of simple functions s_k: $R^{2n} \to C$ such that $s_k \to \varphi$ pointwise. Then $S_k s_k$: $R^{2n} \to \mathcal{H}$ is a sequence of simple functions and pointwise

$$S_k s_k \to e^{i\lambda(rQ+sP)}\phi\varphi(r, s)$$

According to a well-known theorem from the theory of the Bochner integral (see e.g., Blank, Exner, and Havlíček, 1993, Theorem 3.7.4c) the Bochner integral converges if and only if the Lebesgue integral of the norm converges. However,

$$\left\| e^{i\lambda(rQ+sP)}\phi\varphi(r, s) \right\| = \|\phi\| \left|\varphi(r, s)\right| \in L^1(R^{2n})$$

because $\varphi \in L^1(R^{2n})$. That means for a function $\varphi \in L^1(R^{2n})$ the considered Bochner integral converges for arbitrary vector ϕ. Moreover,

$$\|f(Q^\lambda, P^\lambda)\phi\| = \left\|\int e^{i\lambda(rQ+sP)}\phi\varphi(r, s)d^nr\, d^ns\right\|$$
$$\leq \int \|e^{i\lambda(rQ+sP)}\phi\varphi(r, s)\|d^nr\, d^ns = \|\phi\| \int |\varphi(r, s)|d^nr\, d^ns$$

from which according to $\varphi \in L^1(R^{2n})$ the boundedness of the operator $f(Q^\lambda, P^\lambda)$ follows.

So for the construction of the operator $f(Q^\lambda, P^\lambda)$ from (3) the necessary and sufficient condition is $\varphi \in L^1(R^{2n})$. Because f is the inverse Fourier transform of φ, the Riemann–Lebesgue lemma implies the condition $f \in C_\infty(R^{2n})$, which implies

$$\lim_{\|x\|\to\infty} f(x) = 0$$

Therefrom we immediately see that, for instance, the polynomials do not belong among these functions.

We need to calculate

$$h_\lambda(x) = (U_x^\lambda\psi, f(Q^\lambda, P^\lambda)U_x^\lambda\psi)$$

We shall prove in the Appendix the relation

$$h_\lambda(x) = \int (\psi, e^{i\lambda(rQ+sP)}\psi)e^{i(rq+sp)}\varphi(r, s)d^nr\, d^ns$$

The Weyl relations imply that the function $(\psi, e^{i\lambda(rQ+sP)}\psi)$ is continuous in (r, s), so it is measurable. So for each λ the function

$$g_\lambda(r, s) = (\psi, e^{i\lambda(rQ+sP)}\psi)e^{i(rq+sp)}\varphi(r, s)$$

is measurable and

$$|g_\lambda(r, s)| \leq \|\psi\|^2|\varphi(r, s)| = |\varphi(r, s)|$$

Because $\varphi \in L^1(R^{2n})$, we may use the Lebesgue dominated convergence theorem and we have

$$|h_\lambda(x) - f(x)|$$
$$\leq \int |(\psi, e^{i\lambda(rQ+sP)}\psi) - 1||e^{i(rq+sp)}\varphi(r, s)|d^nr\, d^ns$$
$$\leq \int |(\psi, e^{i\lambda(rQ+sP)}\psi) - 1||\varphi(r, s)|d^nr\, d^ns \underset{\lambda\to 0}{\to} 0$$

because

$$\lim_{\lambda\to 0}(\psi, e^{i\lambda(rQ+sP)}\psi) = 1$$

from the Weyl relations. We have then

$$\lim_{\lambda\to 0}(U_x^\lambda\psi, f(Q^\lambda, P^\lambda)U_x^\lambda\psi) = f(q, p)$$

uniformly on R^{2n} if $x = (q, p)$, which completes the proof.

3. THE CONVERGENCE OF DYNAMICS

The symplectic form for the classical projection with the Hamiltonian h_λ is the natural restriction Ω^λ of the canonical symplectic form Ω on the projective Hilbert space $P\mathcal{H}$ to the orbit O_ψ^λ. Bóna (1986) proves that if we multiply the form Ω^λ with the constant $\lambda^2\hbar$, we get exactly the standard symplectic form $dp \wedge dq$ on R^{2n}. From now on the classical projection with the Hamiltonian h_λ will be considered with this standard symplectic form on R^{2n}.

We found out that the Hamiltonians $h_\lambda(x)$ of the classical projections for conveniently chosen functions $f = h$ converge pointwise to the Hamiltonian of the classical limit $h(x)$. There arises a natural question about the convergence of dynamics (time evolutions). Namely the Hamiltonians h_λ and h may be considered to act on the same phase space R^{2n} considered as the Euclidean space, and one can ask whether for the identical initial conditions $x_\lambda(0) = x(0)$ for corresponding time evolutions we will have $x^\lambda(t) \to x(t)$. It can be shown that under certain conditions the answer will be positive.

Theorem 2. Let

$$\frac{\partial h_\lambda}{\partial x_i} \underset{\lambda\to 0}{\to} \frac{\partial h}{\partial x_i} \qquad (i = 1, \ldots, 2n)$$

uniformly on R^{2n}. Then the time evolutions $x^\lambda(t)$ of the systems with Hamiltonian $h^\lambda(x)$ will uniformly converge for $\lambda \to 0$ to the evolution $x(t)$ of the system with the Hamiltonian of the classical limit $h(x)$ on the intervals $\langle 0, t_0\rangle$, where t_0 is arbitrary finite positive, such that all the evolutions exist for $t \in \langle 0, t_0\rangle$ and the initial conditions are $x^\lambda(0) = x(0)$ for all λ sufficiently small.

Proof. For arbitrary small δ_i $(i = 1, \ldots, 2n)$ there exists a μ such that for arbitrary $\lambda < \mu$ the following holds for all $x \in R^{2n}$:

$$\left|\frac{\partial h_\lambda}{\partial x_i} - \frac{\partial h}{\partial x_i}\right| \leq \delta_i$$

Moreover, if $t \leq t_0$ and t_0 is such that the trajectories $x^\lambda(t)$, $x(t)$ all exist for $\lambda < \mu$, then

$$\left\|x^\lambda(t) - x(t)\right\| \leq \sqrt{\sum_{i=1}^{2n} (D_i)^2}$$

where

$$D_i = t_0 \delta_i$$

is the maximum possible deviation between $x^\lambda(t)$ and $x(t)$ in the direction of the ith coordinate axis.

Let t_0 make sense [i.e., there exist the trajectories $x^\lambda(t)$, $x(t)$ for $t < t_0$ where the initial conditions are $x^\lambda(0) = x(0)$]. Let ε be arbitrary, small, and positive. Then we can choose such small numbers δ_i such that for corresponding values D_i,

$$\sqrt{\sum_{i=1}^{2n} (D_i)^2} < \varepsilon$$

holds. According to the previous considerations it is possible to find μ small enough that for $\lambda < \mu$ one has

$$\left|\frac{\partial h_\lambda}{\partial x_i} - \frac{\partial h}{\partial x_i}\right| \leq \delta_i$$

Then for $t \leq t_0$

$$\left\|x^\lambda(t) - x(t)\right\| < \varepsilon$$

which completes the proof.

Now let us find such functions f for which

$$\frac{\partial h_\lambda}{\partial x_j} \underset{\lambda \to 0}{\to} \frac{\partial h}{\partial x_j}$$

uniformly on R^{2n}. We obtain the functions $\partial h_\lambda / \partial q_j$ by a straightforward calculation:

$$\begin{aligned} \frac{\partial h_\lambda}{\partial q_j} &= \lim_{l \to 0} \frac{h_\lambda(q_j + l) - h_\lambda(q_j)}{l} \\ &= \lim_{l \to 0} \int (\psi, e^{i\lambda(rQ+sP)}\psi) \frac{\exp(ir_j l) - 1}{l} e^{i(rq+sp)} \varphi(r, s) d^n r\, d^n s \end{aligned}$$

Using the Lebesgue dominated convergence theorem we can under certain conditions go with the limit behind the integral. The following holds:

$$\lim_{l \to 0} \frac{\exp(ir_j l) - 1}{l} = ir_j$$

Because of

$$\lim_{t \to 0} \left| \frac{e^{it} - 1}{t} \right| = 1, \qquad \lim_{t \to \infty} \left| \frac{e^{it} - 1}{t} \right| = 0,$$

with respect to the continuity of the function $\left|(e^{it} - 1)/t\right|$ the supremum $\sup_{t \in (0,\infty)} \left|(e^{it} - 1)/t\right| = C$ exists. Then

$$\left| \frac{\exp(ir_j l) - 1}{l} \right| < Cr_j$$

so we can use the Lebesgue dominated convergence theorem under the condition $\left|r_j \varphi(r, s)\right| \in L^1(R^{2n})$. Namely

$$\left|(\psi, e^{i\lambda(rQ+sP)}\psi)\right| \left| \frac{\exp(ir_j l) - 1}{l} e^{i(rq+sp)} \varphi(r, s) \right| \leq C \left|r_j \varphi(r, s)\right|$$

so under the assumption of the Lebesgue integrability of the function $\left|r_j \varphi(r, s)\right|$ we have

$$\frac{\partial h_\lambda}{\partial q_j} = \int (\psi, e^{i\lambda(rQ+sP)}\psi) ir_j e^{i(rq+sp)} \varphi(r, s) d^n r \, d^n s$$

Analogously we compute also $\partial h_\lambda / \partial p_j$. It will be sufficient to prove the uniform convergence on R^{2n}

$$\frac{\partial h_\lambda}{\partial x_j} \to \frac{\partial h}{\partial x_j}$$

where analogously to the previous we obtain

$$\frac{\partial h}{\partial x_j} = \int iy_j e^{i(rq+sp)} \varphi(r, s) d^n r \, d^n s$$

The pointwise convergence follows from the Lebesgue theorem. The new variables y_j used in this expression are defined by

$$y_j = r_j, \qquad y_{n+j} = s_j, \qquad i = 1, \ldots, n$$

We show the uniform convergence

$$\frac{\partial h_\lambda}{\partial x_j} \to \frac{\partial h}{\partial x_j}$$

on the whole space (R^{2n}). The following holds:

$$\begin{aligned}\left|\frac{\partial h_\lambda}{\partial x_j}(x) - \frac{\partial h}{\partial x_j}(x)\right| &= \left|\iint ((\psi, e^{i\lambda(rQ+sP)}\psi) - 1) i y_j e^{i(rq+sp)} \varphi(r, s) d^n r\, d^n s\right| \\ &\leq \int \left|(\psi, e^{i\lambda(rQ+sP)}\psi) - 1\right| \left|y_j \varphi(r, s)\right| d^n r\, d^n s\end{aligned}$$

Because of

$$\left|(\psi, e^{i\lambda(rQ+sP)}\psi) - 1\right| \leq 2$$

we may use the Lebesgue theorem and conclude

$$\lim_{\lambda\to 0}\left|\frac{\partial h_\lambda}{\partial x_j} - \frac{\partial h}{\partial x_j}\right| \leq \lim_{\lambda\to 0}\int \left|(\psi, e^{i\lambda(rQ+sP)}\psi) - 1\right|\left|y_j\varphi(r, s)\right| d^n r\, d^n s = 0$$

which ends the proof of the uniform convergence. So we have proved the following result:

Theorem 3. If the Fourier transform φ of the function f satisfies $\varphi \in L^1(R^{2n})$, $y_j\varphi \in L^1(R^{2n})$, $i = 1, 2, \ldots, 2n$, then the dynamics $x^\lambda(t)$ converges uniformly for $\lambda \to 0$ to $x(t)$ on the intervals $\langle 0, t_0\rangle$ for the initial conditions $x^\lambda(0) = x(0)$. In particular, this condition is satisfied for arbitrary $f \in \mathcal{S}(R^{2n})$ (Schwartz space).

Remark. If $f \in \mathcal{S}(R^{2n})$, then $\varphi \in \mathcal{S}(R^{2n}) \subset L^1(R^{2n})$, and at the same time $y_j\varphi \in \mathcal{S}(R^{2n}) \subset L^1(R^{2n})$.

APPENDIX

Let us compute

$$h_\lambda(x) = (U_x^\lambda\psi, f(Q^\lambda, P^\lambda)U_x^\lambda\psi) = (\psi, (U_x^\lambda)^{-1} f(Q^\lambda, P^\lambda)U_x^\lambda\psi)$$

Let us write

$$(U_x^\lambda)^{-1} f(Q^\lambda, P^\lambda) U_x^\lambda = (U_x^\lambda)^{-1} \left(\int e^{i\lambda(rQ+sP)} \varphi(r, s) d^n r\, d^n s \right) U_x^\lambda$$

$$= \int (U_x^\lambda)^{-1} e^{i\lambda(rQ+sP)} U_x^\lambda \varphi(r, s) d^n r\, d^n s$$

where the interchange of the action of the unitaries is possible due to the definition of Bochner integral. Because

$$U_x^\lambda = U_{x/\lambda}$$

it is sufficient to write

$$(U_{x/\lambda})^{-1} e^{i\lambda(rQ+sP)} U_{x/\lambda}$$

We shall use the relation

$$U_{x+y} = \exp\left(\frac{i}{2\hbar}\, x \cdot S \cdot y \right) U_x U_y$$

which is given in Bóna (1986). Therefrom

$$(U_{x/\lambda})^{-1} = \exp\left(\frac{i}{2\hbar} \frac{x}{\lambda} S \left(-\frac{x}{\lambda} \right) \right) U_{-x/\lambda}$$

Moreover

$$e^{i\lambda(rQ+sP)} = \exp\left(\frac{i}{\hbar}\, \hbar\lambda(rQ + sP) \right) = U_{\hbar\lambda(-s,r)}$$

where

$$(-s, r) = (-s_1, \ldots, -s_n, r_1, \ldots, r_n)$$

Then

$$U_{-x/\lambda} U_{\hbar\lambda(-s,r)} = U_{\hbar\lambda(-s,r)} U_{-x/\lambda} \exp\left(\frac{i}{\hbar} (\hbar\lambda)(-s, r) S \left(-\frac{x}{\lambda} \right) \right)$$

because

$$U_x U_y = U_y U_x \exp\left(\frac{i}{\hbar}\, x \cdot S \cdot y \right)$$

So we have

$$\begin{aligned} U_{-x/\lambda}U_{\hbar\lambda(-s,r)} &= U_{\hbar\lambda(-s,r)}U_{-x/\lambda}\exp(i(-s, r)S(-x)) \\ &= U_{\hbar\lambda(-s,r)}U_{-x/\lambda}\exp(i(s, -r)S(q, p)) \\ &= U_{\hbar\lambda(-s,r)}U_{-x/\lambda}e^{i(rq+sp)} \end{aligned}$$

Finally

$$\begin{aligned} &\exp\left(\frac{i}{2\hbar}\left(\frac{x}{\lambda}\right)S\left(-\frac{x}{\lambda}\right)\right)U_{-x/\lambda}U_{x/\lambda} \\ &\quad = \exp\left(\frac{i}{2\hbar}\left(-\frac{x}{\lambda}\right)S\left(\frac{x}{\lambda}\right)\right)U_{-x/\lambda}U_{x/\lambda} = U_0 = I \end{aligned}$$

so we have the result

$$(U_x^\lambda)^{-1}f(Q^\lambda, P^\lambda)U_x^\lambda = \int e^{i(rq+sp)}e^{i\lambda(rQ+sP)}\varphi(r, s)d^nr\ d^ns$$

so

$$h_\lambda(x) = \int (\psi, e^{i\lambda(rQ+sP)}\psi)e^{i(rq+sp)}\varphi(r, s)d^nr\ d^ns$$

where the last step is possible due to the properties of the Bochner integral.

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